Inversion of k-tridiagonal matrices with Toeplitz structure

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Abstract

In this paper, we consider an inverse problem with the k-tridiagonal Toeplitz matrices. A theoretical result is obtained that under certain assumptions the explicit inverse of a k-tridiagonal Toeplitz matrix can be derived immediately. Two numerical examples are given to demonstrate the validity of our results.

Keywords: Inverse; k-tridiagonal matrix; Toeplitz matrix; Euler's formula; Block diagonalizations

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Fast block diagonalization of k-tridiagonal matrices

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Abstract

In the present paper, we give a fast algorithm for block diagonalization of k-tridiagonal matrices. The block diagonalization provides us with some useful results: e. g., another derivation of a very recent result on generalized k-Fibonacci numbers in [M. E. A. El-Mikkawy, T. Sogabe, A new family of k-Fibonacci numbers, Appl. Math. Comput. ٢١٥ (٢٠١٠) ٤٤٥٦-٤٤٦١]; efficient (symbolic) algorithm for computing the matrix determinant.

Keywords: k-tridiagonal matrix; Block diagonalizations; Generalized k-Fibonacci numbers; Determinant; Finite field; General linear group

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A new family of k-Fibonacci numbers

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Abstract
In the present paper, we give a new family of k-Fibonacci numbers and establish some properties of the relation to the ordinary Fibonacci numbers. Furthermore, we describe the recurrence relations and the generating functions of the new family for k = ٢ and k = ٣, and presents a few identity formulas for the family and the ordinary Fibonacci numbers.

Keywords: Fibonacci numbers; Generalized k-Fibonacci numbers; k-Tridiagonal matrix; Determinant; Generating functions; Recurrence relation

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Notes on particular symmetric polynomials with applications

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Abstract
This paper presents some applications using several properties of three important symmetric polynomials: elementary symmetric polynomials, complete symmetric polynomials and the

power sum symmetric polynomials. The applications includes a simple proof of El-Mikkawy conjecture in [M. E. A. El-Mikkawy, Appl. Math. Comput. ١٤٦ (٢٠٠٣) ٧٥٩-٧٦٩] and a very easy proof of the Newton-Girard formula. In addition, a generalization of Stirling numbers is obtained.

Keywords: Elementary symmetric polynomials; Complete symmetric polynomials; Power sum symmetric polynomials; Matrices; Determinant; LU factorization; Stirling numbers

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Generalized harmonic numbers with Riordan arrays

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Abstract
By observing that the infinite triangle obtained from some generalized harmonic numbers follows a Riordan array, we obtain very simple connections between the Stirling numbers of both kinds and other generalized harmonic numbers. Further, we suggest that Riordan arrays associated with such generalized harmonic numbers allow us to find new generating functions of many combinatorial sums and many generalized harmonic number identities. (C) ٢٠٠٧ Elsevier Inc. All rights reserved.

Keywords: Harmonic numbers; Stirling numbers; Riordan array; Symmetric polynomial; Harmonic polynomial; Bernoulli polynomial
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